Lecture 12

Chain Rule in 1-D

Recall the case of a function, $y = g(u)$ and $u = f(x)$. To find $\frac{dy}{dx}$ we used the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

It makes sense that y is a function x since $y = g(u) = g(f(x))$. This means y is totally defined by x.

Chain Rule in n-D

Consider a function, $z = f(x,y)$, with $x = g_1(t)$ & $y = g_2(t)$. Thus we can see $z = f(x,y) = f(g_1(t), g_2(t))$. That is $z = h(t)$.

So, it makes sense that we can write a total derivative of z. That is:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

total

partial

Note the ∂'s and d's.

Consider a function $z = f(x,y)$, with $x = g_1(u,v)$ & $y = g_2(u,v)$. Then we have:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \&$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

So, z has no "total derivative".

Ex 1

If $z = x^2 y + 3xy^4$, $x = \sin(2t)$, & $y = \cos(t)$. Find $\frac{dz}{dt}$ when $t = 0$

$$z \begin{cases} x - t \\ y - t \end{cases}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{dt} = (2x + 3y^4)\,2\cos(2t) + (x^2 + 12xy^3)(-\sin(t))$$

$x(t=0) = 0 \qquad y(t=0) = 1$

$$\frac{dz}{dt} = (3)(2) + (0)(0) = 6$$

Ex. 2

What is the chain rule for the case with $w = f(x, y, z, t)$, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, & $t = t(u, v)$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

Ex 3

If $u = x^4 y + y^2 z^3$, $x = rse^t$, $y = rs^2 e^{-t}$, $z = r^2 s \sin(t)$, find $\frac{\partial u}{\partial s}$ when $r = 2$, $s = 1$, & $t = 0$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial s} = 4x^3 y\left(re^t\right) + \left(x^4 + 2yz^3\right)\left(2rse^{-t}\right) + \left(3y^2 z^2\right)\left(r^2 \sin(t)\right)$$

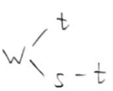
with $r = 2$, $s = 1$, & $t = 0 \rightarrow x = 2$, $y = 2$, $z = 0$

$$\frac{\partial u}{\partial s} = 4(2)^3 2\,(2) + \left(2^4 + 2(2)0\right)\left(2(2)(1)\right) + \left(3(2^2)0\right)\left(2^2(0)\right)$$

$$\frac{\partial u}{\partial s} = 192$$

Ex. 4

With $w = t^2 + \frac{1}{s}$ and $s = t^3 + t$ find $\frac{dw}{dt}$.



$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial s}\frac{ds}{dt} \qquad \frac{dw}{dt} \neq \frac{\partial w}{\partial t}$$

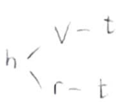
$$\frac{dw}{dt} = 2t + \frac{-1}{s^2}\left(3t^2 + 1\right)$$

$$\frac{dw}{dt} = 2t - \frac{1}{\left(t^3 + t\right)^2}\left(3t^2 + 1\right)$$

Ex. 5

Sand falls in a conical pile at 2π in^3/min. The radius increases at 3 in/min. How fast is the height changing when $h = 10"$ & $r = 8"$

$$V = \frac{1}{3}\pi r^2 h \rightarrow h = \frac{3V}{\pi r^2}$$



$$\frac{dh}{dt} = \frac{\partial h}{\partial V}\frac{dV}{dt} + \frac{\partial h}{\partial r}\frac{dr}{dt}$$

$$\frac{dh}{dt} = \frac{3}{\pi r^2}(2\pi) + \frac{-6V}{\pi r^2}(3)$$

$$h = 10 \quad r = 8, \rightarrow V = \tfrac{1}{3}\pi(8)^2(10)$$

$$\frac{dh}{dt} = \frac{6}{8^2} - 18 \frac{\tfrac{1}{3}\cancel{\pi r^2} h}{\cancel{\pi r^2}}$$

$$\frac{dh}{dt} = \frac{6}{64} - 18\left(\tfrac{1}{3}(10)\right) = \frac{6}{64} - 60$$

Implicit Differentiation

Ex. 6

Let $x^3 + y^3 = 2xy$, find $\frac{dy}{dx}$

Call $w = x^3 + y^3 - 2xy = 0$

$$w \begin{cases} x \\ y - x \end{cases}$$

$$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx} = 0$$

$$-\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y}\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = -\frac{\partial w}{\partial x} \Big/ \frac{\partial w}{\partial y}}$$

$$\frac{dy}{dx} = \frac{-(3x^2 - 2y)}{3y^2 - 2y}$$

$$\frac{dy}{dx} = \frac{2y - 3x^2}{3y^2 - 2y}$$